# Quasi-optimality of approximate solutions in normed vector spaces

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QUASI-OPTIMALITY OF APPROXIMATE SOLUTIONS IN NORMED VECTOR SPACES

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ABSTRACT. We discuss quasi-optimality of approximate solutions to operator equations in normed vector spaces, defined either by Petrov–Galerkin projection or by residual minimization. Examples demonstrate the sharpness of the estimates.

Let $X$ and $Y$ be real normed vector spaces. Let $B : X \to Y'$ be a linear operator. Fix $u \in X$ – the "unknown". Let $X_h \times Y_h \subset X \times Y$ be nontrivial finite-dimensional subspaces. Abbreviate

$$\gamma_h := \inf_{w \in X_h \setminus \{0\}} \|Bw\|_{Y_h'} / \|w\|_X \quad \text{and} \quad |||B||| := \sup_{w \in (u + X_h) \setminus \{0\}} \|Bw\|_{Y_h'} / \|w\|_X. \tag{1}$$

Throughout, we assume the "discrete inf-sup condition": $\gamma_h > 0$. We define $B_h : X_h \to Y_h'$ by $w \mapsto (Bw)|_{Y_h}$. In the first proposition we require $\dim X_h = \dim Y_h$. In the second we admit $\dim Y_h \geq \dim X_h$.

**Proposition 1.** *Suppose* $\dim X_h = \dim Y_h$. *Then there exists a unique* $u_h \in X_h$ *such that*

$$\langle B u_h, v \rangle = \langle B u, v \rangle \quad \forall v \in Y_h. \tag{2}$$

*The mapping* $u \mapsto u_h$ *is linear with* $\|u_h\|_X \leq \gamma_h^{-1} \|Bu\|_{Y_h'}$ *and satisfies the quasi-optimality estimate:*

$$\|u - u_h\|_X \leq (1 + \gamma_h^{-1} |||B|||) \inf_{w_h \in X_h} \|u - w_h\|_X. \tag{3}$$

*Proof.* The map $B_h$ is linear and injective by (1). It is bijective due to finite $\dim X_h = \dim Y_h = \dim Y_h'$. Thus a unique $u_h := B_h^{-1}(Bu)|_{Y_h}$ exists and $u \mapsto u_h$ is linear. By (1), $\gamma_h \|u_h\|_X \leq \|B_h u_h\|_{Y_h'} = \|Bu\|_{Y_h'}$. From $\|u - u_h\|_X \leq \|u - w_h\|_X + \|w_h - u_h\|_X$ and $\gamma_h \|w_h - u_h\|_X \leq \|B(u_h - w_h)\|_{Y_h'} = \|B(u - w_h)\|_{Y_h'} \leq |||B||| \, \|u - w_h\|_{Y_h'}$ we obtain (3). □

**Proposition 2.** *The set* $U_h := \operatorname{argmin}_{w_h \in X_h} \|Bu - Bw_h\|_{Y_h'} \subset X_h$ *of residual minimizers is nonempty, convex and bounded. Any* $u_h \in U_h$ *satisfies the quasi-optimality estimate*

$$\|u - u_h\|_X \leq (1 + 2\gamma_h^{-1} |||B|||) \inf_{w_h \in X_h} \|u - w_h\|_X. \tag{4}$$

*Proof.* The first statement is elementary: consider the metric projection of $(Bu)|_{Y_h} \in Y_h'$ onto $B_h X_h \subset Y_h'$. Quasi-optimality is obtained as above, except that $\|B(u_h - w_h)\|_{Y_h'} \leq \|B(u - u_h)\|_{Y_h'} + \|B(u - w_h)\|_{Y_h'} \leq 2\|B(u - w_h)\|_{Y_h'}$. □

The set $U_h$ of minimizers is a singleton if the unit ball of $Y_h'$ is strictly convex. Since $Y_h$ is finite-dimensional, this is the case if and only if the norm of $Y_h$ is Gâteaux differentiable.

The constants in (3) and (4) are sharp: Take $X = Y = \mathbb{R}^2$ with the $|\cdot|_1$ norm. Then $|\cdot|_\infty$ is the norm of $Y'$. Take $u := (0, 1)$ and $B(w_1, w_2) := (w_1 + w_2, w_2)$. Set $X_h := \mathbb{R} \times \{0\}$ ($\rightsquigarrow B$ is identity on $X_h$). Observe $|||B||| = 1$.

- For (3) let $Y_h := \mathbb{R} \times \{0\}$. Then $\|Bw_h\|_{Y_h'} = \|w_h\|_X$ for all $w_h \in X_h$ gives $\gamma_h = 1$. Now, $u_h = (1, 0) \in X_h$ solves (2). In the quasi-optimality estimate we have $\|u - u_h\|_X = 2$ while $\|u - w_h\|_X = 1$ for $w_h = 0$.
- For (4) let $Y_h := Y$. Again, $\gamma_h = 1$. Since $Bu = (1, 1)$, the set of minimizers $U_h$ is the segment $[0, 2] \times \{0\}$. For $u_h := (2, 0) \in U_h$ we have $\|u - u_h\|_X = 3$ while $\|u - w_h\|_X = 1$ for $w_h = 0$. With a slight perturbation of the norms, say, we can achieve $U_h = \{u_h\}$ without essentially changing the distances.

If $X$ and $Y$ are Hilbert spaces and $B : X \to Y'$ is bounded by $\|B\|$ then in both propositions the mapping $P_h : X \to X$, $u \mapsto u_h$, is a well-defined bounded linear projection with $\|P_h\| \leq \gamma_h^{-1} \|B\|$. The argument of

[1] J. Xu and L. Zikatanov. Some observations on Babuška and Brezzi theories. *Numer. Math.*, 94(1), 2003.

then improves the quasi-optimality estimate to $\|u - u_h\|_X \leq \|P_h\| \inf_{w_h \in X_h} \|u - w_h\|_X$.

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